MATH 579 S26, Exam 2 Solutions

1. Set $a_i = i4^i$. Find the OGF $A(x) = \sum_{i\geq 0} a_i x^i$.

   Note that $i = \binom{i}{1}$ so $\frac{x^1}{(1-x)^2} = \sum_{i\geq 0} ix^i$. Now we substitute $x \mapsto 4x$ to get $\frac{(4x)^1}{(1-4x)^2} = \sum_{i\geq 0} i(4x)^i = \sum_{i\geq 0} i4^i x^i$, so our desired OGF is $A(x) = \frac{4x}{(1-4x)^2}$.

2. Consider the OGF $A(x) = \frac{1-2x^3}{(1-x)^4} = \sum_{n\geq 0} a_n x^n$. Find a closed form for a_n. Simplify for 1pt of extra credit.

   We first split $A(x) = \frac{1}{(1-x)^4} + (-2)\frac{x^3}{(1-x)^4}$. Now, $\frac{1}{(1-x)^4} = \sum_{n\geq 0} \binom{n+3}{n} x^n$. Multiplying by x^3 we get $\frac{x^3}{(1-x)^4} = \sum_{n\geq 0} \binom{n+3}{n} x^{n+3} = \sum_{n\geq 3} \binom{n}{n-3} x^n$ (in the last step we reindexed $n \mapsto n-3$). Multiplying by (-2) and adding we get $A(x) = \sum_{n\geq 0} \binom{n+3}{n} x^n + \sum_{n\geq 3} (-2)\binom{n}{n-3} x^n = \sum_{n=0}^{2} \binom{n+3}{n} x^n + \sum_{n\geq 3} \binom{n+3}{n} - 2\binom{n}{n-3} x^n$. We could now give a_n using cases, but it turns out that $\binom{n}{n-3} = 0$ for $n = 0, 1, 2$, so in fact $A(x) = \sum_{n\geq 0} \binom{n+3}{n} - 2\binom{n}{n-3} x^n$. Hence $a_n = \binom{n+3}{n} - 2\binom{n}{n-3}$, for all $n \geq 0$.

   If we wish to simplify, this equals $\binom{n+3}{3} - 2\binom{n}{3} = \frac{(n+3)(n+2)(n+1) - 2n(n-1)(n-2)}{3!} = \frac{-n^3 + 12n^2 + 7n + 6}{6}$.

3. Let a_i satisfy $a_0 = 1, a_1 = 12, a_i = 6a_{i-1} - 9a_{i-2} + 1$ $(i \geq 2)$. Find the OGF.

   Let $A(x) = \sum_{i\geq 0} a_i x^i$. Multiplying the recurrence relation by x^i and summing over $i \geq 2$ we get $\sum_{i\geq 2} a_i x^i = 6\sum_{i\geq 2} a_{i-1} x^i - 9\sum_{i\geq 2} a_{i-2} x^i + \sum_{i\geq 2} x^i = 6x\sum_{i\geq 2} a_{i-1} x^{i-1} - 9x^2 \sum_{i\geq 2} a_{i-2} x^{i-2} + x^2 \sum_{i\geq 2} x^{i-2}$. Hence $A(x) - a_0 - a_1 x = 6x(A(x) - a_0) - 9x^2 A(x) + \frac{x^2}{1-x}$. Substituting, we get $A(x) - 1 - 12x = 6x(A(x) - 1) - 9x^2 A(x) + \frac{x^2}{1-x}$; rearranging, we get $A(x)(1 - 6x + 9x^2) = 1 + 12x - 6x + \frac{x^2}{1-x} = \frac{(1+6x)(1-x) + x^2}{1-x} = \frac{-5x^2 + 5x + 1}{1-x}$. Finally, we can divide and get $A(x) = \frac{-5x^2 + 5x + 1}{(1-x)(1-6x+9x^2)}$. Or, if you prefer, $A(x) = \frac{-5x^2 + 5x + 1}{(1-x)(1-3x)^2}$.

   If you're curious, the closed form is $a_n = \frac{1 + (38n+9)3^{n-1}}{4}$.

4. Your habit in climbing stairs is to always go up either one or two steps at a time. Going up one step can be with either the left foot or with the right foot. Going up two steps can be with the left foot, the right foot, or jumping with both feet at once. Let $S(n)$ denote the number of ways to reach the n-th stair. Use OGFs to find a closed form for $S(n)$.

   Your eccentric habits satisfy the recurrence relation $S(n) = 2S(n-1) + 3S(n-2)$, for $n \geq 2$. Initial conditions are $S(0) = 1, S(1) = 2$. Set $A(x) = \sum_{n\geq 0} S(n) x^n$. Multiply both sides of the recurrence relation by x^n and summing over $n \geq 2$ we get $\sum_{n\geq 2} S(n) x^n = 2\sum_{n\geq 2} S(n-1) x^n + 3\sum_{n\geq 2} S(n-2) x^n = 2x\sum_{n\geq 2} S(n-1) x^{n-1} + 3x^2 \sum_{n\geq 2} S(n-2) x^{n-2}$. Hence $A(x) - S(0) - S(1)x = 2x(A(x) - S(0)) + 3x^2 A(x)$; rearranging, we get $A(x)(1 - 2x - 3x^2) = S(0) + S(1)x - 2xS(0) = 1$. Hence $A(x) = \frac{1}{1-2x-3x^2} = \frac{1}{(1-3x)(1+x)}$. Now it's partial fraction time. We find $A(x) = \frac{3/4}{(1-3x)} + \frac{1/4}{1+x} = \frac{3}{4}\sum_{n\geq 0} 3^n x^n + \frac{1}{4}\sum_{n\geq 0} (-1)^n x^n = \sum_{n\geq 0} \frac{3}{4} 3^n + \frac{1}{4}(-1)^n x^n$. Hence $S(n) = \frac{3^{n+1} + (-1)^n}{4}$.

5. Let a_n denote the number of ways to color a $1 \times n$ chessboard using the colors red, white, and blue, so that a red square must be followed by a white square. Find the OGF representing this sequence.

   If the first square is red, then the second square must be white, and a_{n-2} colorings of the remainder. If the first square is not red, then there are two choices for that color, and a_{n-1} colorings of the remainder. Hence the recurrence relation is $a_n = 2a_{n-1} + a_{n-2}$ (for $n \geq 2$). We have $a_0 = 1$ and $a_1 = 2$ as initial conditions. Let $A(x) = \sum_{n\geq 0} a_n x^n$. Multiplying our recurrence relation by x^n and summing over $n \geq 2$, we get $\sum_{n\geq 2} a_n x^n = 2\sum_{n\geq 2} a_{n-1} x^n + \sum_{n\geq 2} a_{n-2} x^n = 2x\sum_{n\geq 2} a_{n-1} x^{n-1} + x^2 \sum_{n\geq 2} a_{n-2} x^{n-2}$. Hence $A(x) - a_0 - a_1 x = 2x(A(x) - a_0) + x^2 A(x)$; rearranging, we get $A(x)(1 - 2x - x^2) = a_0 + a_1 x - 2xa_0 = 1$. Hence $A(x) = \frac{1}{1-2x-x^2}$.

   Fun fact: the closed form is $a_n = \frac{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}}{2\sqrt{2}}$. These are called Pell numbers (shifted by one).

6. Let $a \in \mathbb{N}_0$. Prove that $\sum_{i\geq 0}\binom{a}{2i} = \sum_{i\geq 0}\binom{a}{2i+1}$.

Just one proof on this exam!

The binomial theorem says that $(1+x)^a = \sum_{i\geq 0}\binom{a}{i}x^i$. Substituting $x=-1$, we get $0 = \sum_{i\geq 0}\binom{a}{i}(-1)^i = \sum_{i\geq 0}\binom{a}{2i}(-1)^{2i} + \sum_{i\geq 0}\binom{a}{2i+1}(-1)^{2i+1} = \sum_{i\geq 0}\binom{a}{2i} - \sum_{i\geq 0}\binom{a}{2i+1}$. Rearranging, the desired result follows.

7. Use the multinomial theorem to count the number of distinct rearrangements of Ponomarenko.

In general, for $a \in \mathbb{N}_0$, we consider $(x_P + x_o + x_n + x_m + x_a + x_r + x_e + x_k)^a$. We seek the coefficient of $x_P^1 x_o^3 x_n^2 x_m^1 x_a^1 x_r^1 x_e^1 x_k^1$, which the multinomial theorem tells us is $\frac{11!}{1!3!2!1!1!1!1!1!} = \frac{11!}{12}$. If you're curious, that's $3,326,400$ (though you don't need to compute this by hand).

8. Consider the sequence $a_0 = 1$, $a_n = na_{n-1} + n(n-1)$ (for $n \geq 1$). Find the EGF $A(x)$.

Set $A(x) = \sum_{i\geq 0}\frac{a_n}{n!}x^n$. Multiplying both sides of the recurrence relation by $\frac{x^n}{n!}$ and summing over $n \geq 1$, we get $\sum_{n\geq 1}\frac{a_n}{n!}x^n = \sum_{n\geq 1}\frac{na_{n-1}}{n!}x^n + \sum_{n\geq 1}\frac{n(n-1)}{n!}x^n$. Note that $\frac{1(1-0)}{1!} = 0$, so that last sum might as well be $n \geq 2$. We continue as $= x\sum_{n\geq 1}\frac{a_{n-1}}{(n-1)!}x^{n-1} + x^2\sum_{n\geq 2}\frac{1}{(n-2)!}x^{n-2}$. Hence $A(x) - a_0 = xA(x) + x^2e^x$; rearranging we get $A(x)(1-x) = a_0 + x^2e^x = 1 + x^2e^x$. Hence $A(x) = \frac{1+x^2e^x}{1-x}$.

Fun fact: it's not too hard to show that $\frac{a_n}{n!} = 1 + \sum_{i=0}^{n-2}\frac{1}{i!}$, which approaches $1+e$ as $n \to \infty$. Hence $a_n \to n!(1+e)$.

9. Let a_n denotes the number of ways of splitting a list of n people (say, in alphabetical order), into sublists of size 4 or 5. Each sublist must consist of consecutive names. Find an appropriate generating function (either OGF or EGF) and use this to calculate a_{17}.

Set $C(x) = x^4 + x^5$, the OGF satisfying $c_4 = c_5 = 1$ (and zero otherwise). By the OGF Composition Theorem, the number of ways to split $[n]$ into disjoint nonempty subintervals, and then do c on each one (i.e. that subset is of size 4 or 5) is given by $A(x) = \frac{1}{1-C(x)} = \frac{1}{1-x^4-x^5} = \sum_{n\geq 0}(x^4+x^5)^n$. From this we seek the coefficient of x^{17}. It turns out that the only way to get x^{17} is from $n=4$, since $(x^4)^3(x^5)^1 = x^{17}$, and no other combination of 4's and 5's gives 17. We turn now to $(x^4+x^5)^4 = x^{16}(1+x)^4$. The coefficient of x^1 in $(1+x)^4$ we find with the Binomial Theorem; it is $\binom{4}{1} = 4$. Putting it all together, we find $a_{17} = 4$.

Note: this corresponds to the four positions that the 5-sublist can be in, among the four sublists.

10. Let b_n denote the number of ways of partitioning a set of n people into groups of size 4 or 5. Find an appropriate generating function (either OGF or EGF) and use this to calculate b_{17}.

Set $C(x) = \frac{x^4}{4!} + \frac{x^5}{5!}$, the EGF satisfying $c_4 = c_5 = 1$ (and zero otherwise). By the EGF Composition Theorem, the number of ways to split $[n]$ into disjoint nonempty subsets, and then do c on each one (i.e. that subset is of size 4 or 5) is given by $B(x) = e^{C(x)} = e^{\frac{x^4}{4!}+\frac{x^5}{5!}} = \sum_{n\geq 0}\frac{1}{n!}\left(\frac{x^4}{4!}+\frac{x^5}{5!}\right)^n$. From this we seek the coefficient of x^{17}. It turns out that again the only way to get x^{17} is from $n=4$, since $(x^4)^3(x^5)^1 = x^{17}$, and no other combination of 4's and 5's gives 17. We turn now to $\frac{1}{4!}\left(\frac{x^4}{4!}+\frac{x^5}{5!}\right)^4 = \frac{x^{16}}{(4!)^5}(1+\frac{x}{5})^4$. The coefficient of x^1 in $(1+\frac{x}{5})^4$ we find with the Binomial Theorem; it is $\binom{4}{1}(\frac{1}{5})^1 = \frac{4}{5}$. Putting it all together, we find $b_{17} = \frac{17!}{(4!)^5}\frac{4}{5}$. If you're curious, this equals $35,735,700$.